Analytical formulation of the error probability of a QPSK transmission impaired by the joint action of gaussian and impulse noises

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ABSTRACT

Based on previously published theories, this paper establishes the analytical formulation of the mean bit error probability of a QPSK (Quadrature Phase Shift Keying) modulated signal transmitted on a xDSL (Digital Subscriber Line) channel impaired by both gaussian and impulse noises.

KEY WORDS: modelling and simulation, noise statistics, impulse noise, bit error rate.

1. INTRODUCTION

Impulse noise is known to be the most degrading factor in Digital Subscriber Line (xDSL) technologies. It is a non stationary crosstalk issued from temporary electromagnetic events in the vicinity of telephone lines. Induced voltage can be as high as 100 mV and typically lasts from 10 to 100 microseconds.

Since the 60's, efforts have been made in order to understand and model impulse noise on copper pairs. One of the well known models in a representative single pulse, also named 'Cook Pulse' [1], especially used for modelling purposes in the HDSL bandwidth. Unfortunately, such a representative pulse is not able to describe the statistical properties of the noise. Nowadays, a large amount of work (measurement and simulation methods of impulse noise) is being realized, mainly in the frame of the VDSL normalization process (cf. documents of the TM6 working group of ETSI, i.e. [2]).

To our best knowledge, the most complete measurement campaign and the resulting analytical model of impulse noise statistics have been published by Deutsche Telekom workers in 1993/94 [3]. They carried out an extensive measuring campaign at central offices in a 4.25 MHz bandwidth. It led to an impulse noise model incorporating the mean Power Spectral Density (PSD), the phase properties as well as the statistics of inter-arrival times, lengths and amplitudes. Based on this model, a team of the Centre for Wireless Communications in Singapore defined some analytic relations. They first expressed the mean bit error rate of a binary antipodal data transmission only impaired by impulse noise [4]. Secondly, in the frame of I & Q demodulation, they extended the probability density function (PDF) of the noise voltage to a narrowband impulse noise amplitude, resolved into in-phase and quadrature components, each being identically and independently modelled by a Generalised Error Distribution (GED) [5].

In this paper, we use previous theories (cf. [3], [4] and [5]) to establish the analytical formulation of the mean bit error probability of a QPSK transmission impaired by the common action of gaussian noise and impulse noise having the properties described in [3]. Section 2 explains the used impulse and gaussian noises characteristics, section 3 gives the GED parameters and section 4 details the mathematical process to obtain the wanted analytical formula. Section 5 concludes this paper.

2. DESCRIPTION OF THE RESULTS OF THE GERMAN FIELD TRIAL [3]

A. Description of the field trial

From 1992 to 1994, the Deutsche Bundespost recorded a huge amount of data on six sites of their telephone network (50.000 records per location): Darmstadt, Ober-Ramstadt, Frankfurt OVSt 72, Kassel, Berlin DA 226 and Mainz. The measured lines were chosen to be heavily impaired by impulse noise. Between impulses, noise was considered to be white gaussian noise. The setup consisted in a digital oscilloscope triggered when an impulse event was present. The measurement bandwidth was limited to 4.25 MHz by a low pass filter and the sampling frequency (f_s) was fixed to 10.24 MHz. As results, data where transformed into histograms of voltages. A 32 bits counter was used in the setup in order to build impulse duration and inter-arrival time histograms. Starting from those histograms, the authors of [3] have defined Probability Density Functions (PDF), fitting all experimental sites, only with varying values of

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the PDF parameters. The properties of the Power Density Spectrum (PDS) of impulses as well as their phase properties were also modelled but are of little interest in this paper.

B. *PDF of voltages of impulsive noise samples*

The approximation of the pure impulsive noise PDF of the voltage u was found by the authors to be a stretched exponential. It uses a single parameter u_0, being a positive location dependent parameter expressed in volts:

$$f_i(u) = \frac{e^{-|u/u_0|^{1/5}}}{240 u_0} \qquad (1)$$

In order to take into account the white gaussian noise, which is the only disturbance present between the impulses, the following weighted sum must be used :

$$f_{tot}(u) = N f_n(u) + (1-N) f_i(u) * f_n(u) \qquad (2)$$

where $f_n(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}}$ (3)

is the PDF of the white gaussian noise, with the standard deviation σ, and N ∈ [0 1] is a coefficient of proportionality. During an impulse, white gaussian noise is also present. Therefore, assuming statistical independence between both processes, the addition of the two disturbances results in the convolution of their PDF. Experimental parameters are :

Location	N	σ [mV]	u_0 [nV]
Darmstadt	0.991	0.19	0.7
Ober-Ramstadt	0.999	0.54	20.9
Frankfurt OVSt72	0.947	0.55	46.6
Kassel	0.946	-	1.4
Berlin DA 226	0.706	0.60	123.1
Mainz	0.996	-	18.2

Table 1: Parameters of the PDF of voltages [3].

C. *PDF of inter-arrival times*

The inter-arrival times were calculated as the difference of the recorded counter outputs at trigger instants. The authors found that the PDF of inter-arrival times could not be modelled by the usual Poisson process, because the mean impulse occurrence, which depends on the calling rate, varies during the measurement duration. The solution was to model the PDF only for daytime, as its statistics are more severe and closer to human activity. The worst case approximation was achieved by a generalization of the Poisson law in a logarithmic representation :

$$f_d(\xi) = \frac{10^{a_1}}{\ln(10)} \xi^{(a_4-1)} 10^{-\frac{a_4}{\ln(a_2)}(a_2)^{\log_{10}(\xi)-a_3}} \qquad (4)$$

where ξ = t/100 ns, a_1 is a normalization constant, a_2 defines the radius of the curvature of the PDF at the maximum, a_3 determines the position of the maximum and a_4 specifies the slope of the curve at its beginning. Experimental parameters are :

Location	a_1	a_2	a_3	a_4
Darmstadt	-4.60	1.68	6.70	0.88
Ober-Ramstadt	-3.26	6.38	5.43	0.62
Frankfurt OVSt72	-2.66	11.17	5.00	0.53
Kassel	-3.73	5.04	5.62	0.70
Berlin DA 226	-7.85	2.80	5.04	1.91
Mainz	-5.14	2.22	5.15	1.26

Table 2: Parameters of the PDF of inter-arrival time [3].

From equation (4), it is possible to calculate the analytical expression of the mean inter-arrival time between impulses that can be expressed as :

$$\lambda = \frac{\ln(10)\log_{10}(a_2)}{10^{a_1}\,\Gamma\!\left(\frac{a_4+1}{\log_{10}(a_2)}\right)\left[\ln\left(10^{\frac{a_4}{(a_2)^{a_3}\ln(a_2)}}\right)\right]^{-\frac{a_4+1}{\log_{10}(a_2)}}} \qquad (5)$$

D. *PDF of impulse lengths*

The authors of [3] found that the PDF of the length of impulses can be approximated by the sum of two weighted log-normal functions :

$$f_l(t) = B\frac{1}{s_1 t\sqrt{2\pi}} e^{-\frac{1}{2s_1^2}\ln^2(t/t_1)} + (1-B)\frac{1}{s_2 t\sqrt{2\pi}} e^{-\frac{1}{2s_2^2}\ln^2(t/t_2)} \qquad (6)$$

where t_1, t_2 are the median values and s_1, s_2 are the shape parameters of the log-normal densities. Experimental parameters are :

Location	B	$\frac{s_1 f_s}{10^6}$	$\frac{s_2 f_s}{10^6}$	t_1 [μs]	t_2 [μs]
Darmstadt	0.166	8.26	8.96	4.68	128.2
Ober-Ramstadt	0.660	14.54	1.71	109.26	107.1
Frankfurt OVSt72	0.940	9.83	1.12	93.10	77.8
Kassel	0.236	6.28	8.14	14.50	108.2
Berlin DA 226	0.060	7.68	10.44	6.85	158.2
Mainz	0.256	7.68	10.63	7.95	125.7

Table 3: Parameters of the PDF of the impulse lengths[3]

From equation (6), it is easy to calculate the analytical expression of the mean impulse duration, that is (with $\mu_i = \ln(t_i f_s)$ and $\sigma_i = s_i / f_s$):

$$\bar{\tau} = B e^{\left(\mu_1 + \frac{\sigma_1^2}{2}\right)} + (1-B) e^{\left(\mu_2 + \frac{\sigma_2^2}{2}\right)} \qquad (7)$$

2. DESCRIPTION OF THE BI-DIMENSIONNAL GED (GENERALISED ERROR DISTRIBUTION) USED FOR IN-PHASE AND IN-QUADRATURE IMPULSE NOISES REPRESENTATION [5]

For high speed data transmission, M-QAM modulation (M = 4 for a QPSK – Quadrature Phase Shift Keying - transmission) is a potential candidate as it provides a good spectral efficiency. At the receiver side (cf. Figure 1), the signal and the noise encounter an I & Q demodulation.

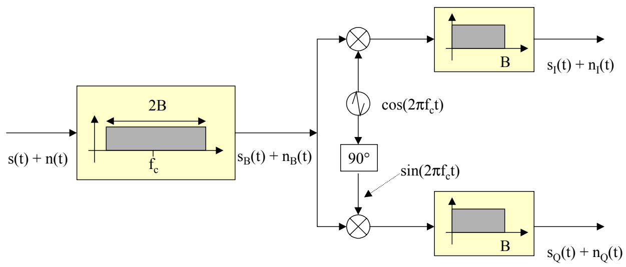

Figure 1: Principle of QPSK transmission and demodulation.

Therefore, bandpass noise $n_B(t)$ at the receiver input is expressed as:

$$n_B(t) = n_I(t)\cos(\omega_c t) - n_Q(t)\sin(\omega_c t) \quad (8)$$

where, $n_I(t)$ and $n_Q(t)$ are respectively the lowpass I & Q components and ω_c is the carrier pulsation.

Based on the experimental results of [3] (previously described in section 2), the authors of [5] found an appropriate impulse noise model to characterize two-dimensional modulations. They took a reasonable assumption that is, as far as the impulse noise duration and inter-arrival time are concerned, their PDF (cf. equations (6) and (4)) are unaltered by the receiver filtering and by the demodulation processes. Therefore, given the expression of equation (1) as the baseband impulse noise PDF, the authors of [5] proved that I & Q PDF are distributed following a Generalized Error Distribution (GED) [7], defined as:

$$f^{(p)}(u) = \frac{1}{2\, w_0\, p^{((1/p)-1)}\Gamma(1/p)} e^{\left(-\frac{1}{p}\left|\frac{u}{w_0}\right|^p\right)} \quad (9)$$

where $\Gamma(*)$ is the Gamma function, p is the order of the GED distribution[1] and $w_0 > 0$ is a scale parameter related to the variance of the distribution. In the case of Mainz field trial results (cf. [3] and Table 1), GED parameters are:

	Bandwidth = 0.25 MHz	Bandwidth = 0.5 MHz
p	1/3	1/4
w_0	0.03	0.02

Table 4: GED parameters for Mainz field trial [5].

[1] p = 1 is the Laplacian distribution and p = 2 is the Gaussian distribution.

The PDF of impulse noise's amplitude are assumed to be identical and independent between I & Q components. So, the joint probability density function is the product of $n_I(t)$ and $n_Q(t)$ and looks like the graph shown on Figure 2.

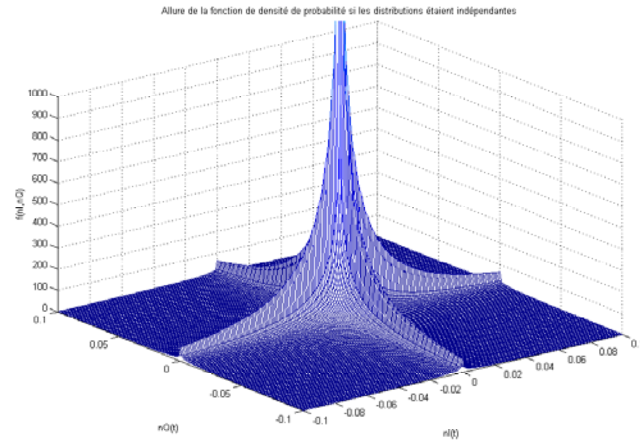

Figure 2: Joint probability density function of the impulse noise amplitude of I and Q branches

3. JOINT IMPACT OF GAUSSIAN NOISE AND IMPULSE NOISE ON QPSK SIGNALS

A. Introduction

In this paper, we assume, like in [6], that the gaussian noise contribution is negligible during the impulse noise duration. The modulation used is a Gray coded QPSK signal whose constellation diagram looks like Figure 3.

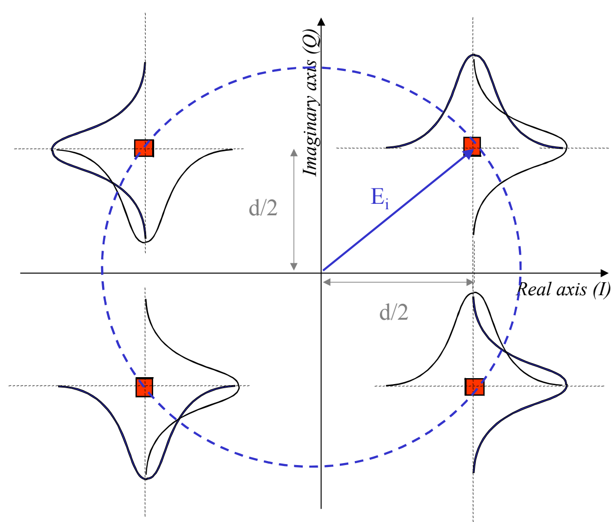

Figure 3: Constellation diagram of a QPSK signal with the schematic of I & Q dispersions of received symbols around the expected symbol.

The symbol error probability P_{SE} can be defined as the complement to the probability of receiving a correct signal P_c. In the case of the QPSK signal of the latter figure, P_c is the probability of interpreting all received symbols in their good quadrant of the constellation diagram, that is:

$$P_{SE} = 1 - P_c = 1 - \sum_{i=1}^{M} P(S_i).P(C|S_i) \qquad (10)$$

where $P(S_i)$ is the occurrence of symbol S_i (1/4 for QPSK), $P(C|S_i)$ is the probability of correctly receiving the symbol S_i, i.e. the probability that the noise corrupting the transmission is localized in the quadrant of the expected symbol:

$$P(C|S_i) = P\left(-\frac{d}{2} < n_I < +\infty\right) P\left(-\frac{d}{2} < n_Q < +\infty\right) = r^2 \qquad (11)$$

where n_I and n_Q are the random variables of the induced noise on I and Q components.

From Figure 3, it is also possible to deduce the expression of the average power associated to a symbol, that is:

$$\overline{E} = \frac{1}{M} \sum_{i=1}^{M} E_i = \frac{d^2}{2} \qquad (12)$$

B. Calculation of the joint impact of gaussian noise and impulse noise

In this case, considering the previous assumption, the total probability density function (PDF) (cf. equation (2)) on I or on Q axis is given by:

$$f_{tot}^{(p)}(u) = (1 - \lambda\overline{\tau}) f_n(u) + \lambda\,\overline{\tau}\, f^{(p)}(u) \qquad (13)$$

where $\lambda\overline{\tau}$ represents the fraction of time when the impulse noise is present (cf. equations (5) and (7)), $f_n(u)$ is equation (3) and $f^{(p)}(u)$ is formula (9). Therefore,

$$f_{tot}^{(p)}(u) = (1 - \lambda\overline{\tau}) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} + \lambda\,\overline{\tau}\,\gamma e^{-\varepsilon|u|^{\frac{1}{\delta}}} \qquad (14)$$

With $\gamma = \frac{1}{2\, w_0\, p^{((1/p)-1)}\Gamma(1/p)}$, $\varepsilon = \frac{1}{p|u_0|^p}$ and $\delta = \frac{1}{p}$

By definition, $r = \int_{-d/2}^{+\infty} f_{tot}^{(p)}(u)du$. Therefore,

$$r = \underbrace{(1 - \lambda\overline{\tau}) \frac{1}{\sigma\sqrt{2\pi}} \int_{-d/2}^{+\infty} e^{-\frac{u^2}{2\sigma^2}} du}_{I} + \underbrace{\lambda\,\overline{\tau} \int_{-d/2}^{+\infty} \gamma e^{-\varepsilon|u|^{1/\delta}} du}_{II} \qquad (15)$$

• Firstly, we calculate the expression of term I.

$$I = (1 - \lambda\overline{\tau}) \frac{1}{2} \operatorname{erfc}\left(\frac{-d/2}{\sigma\sqrt{2}}\right) \qquad (16)$$

• Secondly, we calculate the expression of term II.

$$II = \lambda\overline{\tau}\gamma \left[\int_0^{d/2} e^{-\varepsilon|u|^{1/\delta}} du + \int_0^{+\infty} e^{-\varepsilon|u|^{1/\delta}} du \right]$$

By imposing as a change of variable $z = |u|^{1/\delta}$, it comes:

$$II = \lambda\overline{\tau}\gamma\delta \left[\frac{e^{-\varepsilon\left(\frac{d}{2}\right)^{\frac{1}{\delta}}}}{-\varepsilon} \left(\left(\frac{d}{2}\right)^{\frac{\delta-1}{\delta}} - \frac{(\delta-1)\left(\frac{d}{2}\right)^{\frac{\delta-2}{\delta}}}{-\varepsilon} + \frac{(\delta-1)(\delta-2)\left(\frac{d}{2}\right)^{\frac{\delta-3}{\delta}}}{(-\varepsilon)^2} - \ldots + (-1)^{\delta-1} \frac{(\delta-1)!}{(-\varepsilon)^{\delta-1}} \right) - (-1)^{\delta-1} \frac{(\delta-1)!}{(-\varepsilon)^{\delta}} + \frac{\Gamma(\delta)}{\varepsilon^{\delta}} \right] \qquad (17)$$

Therefore, we have r = I + II,

And $P_{SE} = 1 - 4\frac{1}{4}r^2 = 1 - r^2$.

• It is now necessary to link P_{SE} with the SNR to see how it evolves with the mean total noise power.

SNR is defined by the following equation:

$$SNR = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right) \qquad (18)$$

Taking this equation as a basis, we need to calculate the total variance.

For the gaussian noise part, the situation is clear as its variance is $(1 - \lambda\overline{\tau})\sigma^2$. For the impulse noise portion, we proceed to the same change of variable $z = |u|^{1/\delta}$ and write:

$$\sigma_{GED}^2 = \int_{-\infty}^{+\infty} u^2 \gamma e^{-\varepsilon|u|^{\frac{1}{4}}} du = 2\gamma\delta \int_0^{+\infty} z^{3\delta-1} e^{-\varepsilon z} du = 2\gamma\delta \frac{\Gamma(3\delta)}{\varepsilon^{3\delta}} \qquad (19)$$

Therefore, $SNR = 10\log_{10} \frac{\overline{E}}{2\left[(1 - \lambda\overline{\tau})\sigma^2 + \lambda\overline{\tau}\sigma_{GED}^2\right]}$,

and $\overline{E} = 10^{\frac{SNR}{10}}\, 2\left[(1 - \lambda\overline{\tau})\sigma^2 + \lambda\overline{\tau}\sigma_{GED}^2\right]$. $\qquad (20)$

With $d = \sqrt{2\overline{E}}$ (cf. equation (12)), we obtain the desired link between the mean symbol error probability P_{SE} of

QPSK transmissions jointly corrupted by gaussian and impulse noise, and the SNR.

Having assumed a Gray coding, the mean bit error rate probability is $P_e = \frac{1}{\log_2(M)} P_{SE} = \frac{1}{2} P_{SE}$ (21)

C. Results

In order to compute formula (21) the product $\lambda\bar{\tau}$ has to be evaluated. We used equations (5) and (7) to estimate the mean probability of presence of an impulse. Table 5 shows the results (columns 2 and 3 are rounded):

Location	λ	$\bar{\tau}$	$\lambda\bar{\tau}$
Darmstadt	$0.04\ 10^{-6}$	$1.62\ 10^{3}$	$0.071616\ 10^{-3}$
Ober-Ramstadt	$3.20\ 10^{-6}$	$2.40\ 10^{3}$	$7.6926\ 10^{-3}$
Frankfurt OVSt72	$10.33\ 10^{-6}$	$1.47\ 10^{3}$	$15.1680\ 10^{-3}$
Kassel	$1.90\ 10^{-6}$	$1.20\ 10^{3}$	$2.2286\ 10^{-3}$
Berlin DA 226	$6.85\ 10^{-6}$	$2.57\ 10^{3}$	$17.5720\ 10^{-3}$
Mainz	$3.8\ 10^{-6}$	$1.67\ 10^{3}$	$6.3374\ 10^{-3}$

Table 5: Parameters of the mean impulse occurrence.

On the one hand, as Table 1 shows, there is no information on the white gaussian noise standard deviation for the Mainz location. On the other hand, as Table 4 shows, there exits only GED information for the Mainz location. Therefore, looking on the impulse probability of occurrence at Table 5, we decided, although it has no physical meaning, to use the standard deviation value of the Ober-Ramstadt location to illustrate our theory as Mainz and Ober-Ramstadt $\lambda\bar{\tau}$ products are of the same order of magnitude. Figure 4 shows the result of the computation of equation (21) for 0.5 MHz and 0.25 MHz bandwidths (cf. Table 4).

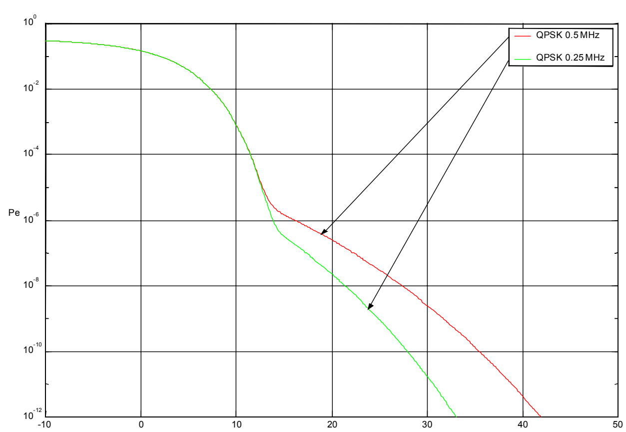

Figure 4: Comparison of the mean bit error probability in the case of a QPSK signal (bandwidth are 0.5 MHz and 0.25 MHz).

On the right hand side of the figure, we see that the computed bit error rate of the QPSK in the case of the 0.5 MHz bandwidth is worse than in the case of a 0.25 MHz bandwidth for a same SNR value. This phenomenon is due to the fitted parameters characterizing the used GED (cf. [5] and Table 4). The explanation is that the GED distribution in the case of the 0.5 MHz bandwidth is more spread than in the case of the 0.25 MHz bandwidth. It allows therefore larger output noise amplitude values samples, which have a more severe impact on the transmission performance than in the other case.

4. CONCLUSION

Based on experimental results and theories developed in [3], [4] and [5], an analytical procedure has been established to calculate the mean impact on bit error rate probability of the joint action of white gaussian noise and impulse noise on xDSL transmissions. It has been applied to QPSK modulation, assuming successively that the impact of the white gaussian noise is negligible during impulse noise, and that gaussian and impulse noise are not correlated, as well as I & Q noise distributions. Those established formulas can now be used to estimate mean data transmission performance over digital subscriber lines.

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