One-dimensional wrinkling of thin membranes

Physical experiments [3]

A film, initially flat, is laid on a substrate such as water and compressed horizontally at both edges. It first starts to wrinkle, taking a sinusoidal form, but, for larger compressions, seems to concentrate at the centre.

Steps leading to the limit problem \( \delta \to 0 \):
1. Boundedness of \( \{ u(\delta) \} \) for any \( (u_0)_{\delta \to 0} \) minisizers:
2. Inequality \( \| u(\delta) \| \leq \| u \| \) for all \( \delta \approx 0 \):
3. The sequence \( (u(\delta)/\sqrt{\delta}) \) is also bounded in \( H^1_0 \):
4. Let us call \( \alpha_1 \) and \( \beta_1 \) the two Lagrange multipliers appearing in the Euler-Lagrange equation (see (3)) by a good choice of “test functions” \( \delta \), we can deduce estimates on these constants, permitting us to divide this equation by \( \sqrt{\delta} \), giving the limit case (4).

The equation (5) admits nontrivial solutions only if \( \alpha < \sqrt{\delta} \), the characteristic polynomial then has two pairs of purely imaginary complex roots. For convenience reasons, let us denote their moduli by \( \mu \) and \( \nu \) with \( \mu > \nu \).

By successively imposing the boundary conditions on a general solution \( \nu \), we get the two equations in \( \mu \) and \( \nu \) for which a part of the solutions is drawn.

1D or 2D space of solutions for equation (5)?

\( 2D \) space \( (\mu, \nu) \) in intersection with \( \{ (\mu, \nu) \} \in \mathbb{N} \times \mathbb{N} \) with \( \mu + \nu \) even.

\( 1D \) space \( (\mu, \nu) \) in (only) one curve.

What about eigenvalues? We get \( \alpha = \sqrt{\delta}(\mu + \nu) \). Thus, for fixed \( K \), the lowest eigenvalue \( -\alpha \) is found for \( (\mu, \nu) \) lying on the curve closest to the diagonal (depending on \( K \), it is curve n’1 or n’2).

Possible degeneracy of eigenvalues: e.g. for the first one, for all \( k \in \mathbb{N} \), \( K = \frac{\pi^2}{4}(k + 1)^2 \) will give a 2D first eigenspace.

Some properties on the “limit solutions”

- Form of the solutions: We have \( u(\delta) = \delta (\xi - \frac{1}{2}) \) with, in case 1D space,
  \[ w(t) = A \left( \cos(\mu t) \cos(\pi \nu t) - \cos(\nu t) \cos(\pi \mu t) \right) \]
  \[ w(t) = A \left( \sin(\mu t) \sin(\nu t) - \sin(\nu t) \sin(\mu t) \right) \]
  where \( A \) is determined (up to sign) by (6).
- In case of 2D space, \( w(t) \) can be any linear combination of two functions like \( \cos(\mu t) \) and \( \cos(\nu t) \).
- Symmetry: if \( \mu(\delta) \) is a solution of (5) then \( \mu(L - \delta) \) is also solution; it is \( \pm w(s) \) in case 1D-space and can be described from \( w(s) \) in case 2D-space.
- Parity of all solutions in case 1D-space: oddness on red curves, evenness on blue ones.
- Number of roots depending on \( \mu \) and \( \nu \): in case 1D-space,
  - At least \( n - 1 \) inner roots when being on the \( n \)-th curve.
  - Increases by 2 after a crossing for the curve going below. It seems that each newly created root comes from an edge.
  - All roots are simple. This behaviour is found (at least numerically) also for solutions living in the 2D spaces.

Future work

- Evolution of the “degeneracy points” (2D space) when \( \delta \) grows.
- Study of the equation as \( K \to 0 \) (related to the Euler-Bernoulli elastica problem).
- Continuation algorithm applied to elastica curves.
- Study of the equation for large values of \( K \).
- Variable coefficients in equation (5).

We assume that there is no variation other than those in the compression direction ⇒ one-dimensional parametrization. Minimization of the energy due to:
- folds (which is measured by its curvature);
- potential energy due to the displacement of the substrate underneath.
⇒ We are seeking for minima of the functional
\[ E : X \to \mathbb{R} : u \to \frac{1}{2} \int_0^L |u'|^2 \, ds + \frac{1}{2} \int_0^L (y_\delta(s))^2 \cos u(s) \, ds \]
where \( y_\delta(s) = \int_0^s \sin u(t) \, dt \) is a constant relative to the substrate and \( X \) describes the space of admissible functions:
\[ X = \{ u \in H^1_0(0, L) ; \mathbb{R} \mid \int_0^L \cos u(s) \, ds = L - \delta \text{ and } \int_0^L \sin u(s) \, ds = 0 \} \]

Going to the limit \( \delta \to 0 \):

Euler-Lagrange equation of the problem:
\[ \frac{d\mathcal{E}(\mu, \nu)}{d\delta} \cdot v + \alpha \int_0^L \sin u \cdot v + \beta \int_0^L \cos u \cdot v = 0 \]
As \( \delta \to 0 \), we obtain:
\[ \int_0^L u''v'' + KL \int_0^L u' + \alpha \int_0^L u'v' + \beta \int_0^L v' = 0 \]
where \( \alpha = \lim \alpha_1 \) and \( \beta = \lim \beta_1 \sqrt{\delta} \) (up to subsequences).
\[ \int_0^L w^2 = 2. \]

Numerical experiments: continuation algorithm applied to the “limit solutions”
Plot of the energies for the first two solutions for \( K = \frac{2 \pi^2}{L^2} \) with, respectively \( k = 0.01, 2, 3 \). Below are the curves obtained by continuation, followed by the resulting curves (x, y). (Length L is fixed to 10 and \( \delta \) varies from 0.05 to 3.5 by step 0.05.)

References